Schnorr triviality is equivalent to being a basis for tt-Schnorr randomness

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Overview

	Schnorr random	tt-Schnorr random
trace	comp. traceable	comp. tt-traceable (FS2010)
low	low for SR	low for tt-SR (FS2010)
low	low for c.m.m.	low for t.m.m. (M2011)
trivial	Schnorr trivial	
trivial	wdm-reducible to 0	
basis		basis for tt-SR

Lowness notions for tt-Schnorr randomness

Four equivalent notions

Theorem (Nies 2005, Hirschfeldt-Nies-Stephan 2007)
The following are equivalent for a set A:

1. A is low for ML-randomness,
2. A is low for K,
3. A is K-trivial,
4. A is a basis for ML-randomness.

Schnorr randomness version

Similarly we can define

1. lowness for Schnorr randomness,
2. Schnorr triviality.

Theorem (Downey Griffiths and LaForte)
There is a Turing complete Schnorr trivial c.e. set.

Hence, lowness for Schnorr randomness is not equivalent to Schnorr triviality.

Schnorr triviality & tt-reducibility

Theorem (Franklin and Stephan 2010)

The following are equivalent for a set A:

1. A is low for tt-Schnorr randomness.
2. A is computably tt-traceable,
3. A is Schnorr trivial,

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tt-Schnorr randomness

X is A-Schnorr random if $d(X \upharpoonright n) \leq h(n) + O(1)$ for all A-computable orders h and martingales $d \leq_{\mathrm{T}} A$.

Definition (Franklin and Stephan 2010)
X is A-tt-Schnorr random if $d(X \upharpoonright n) \leq h(n) + O(1)$ for all computable orders h and martingales $d \leq_{\mathrm{tt}} A$.

Definition (Franklin and Stephan 2010)
A is low for tt-Schnorr randomness if each Schnorr random set is A-tt-Schnorr random.

Low for tt-reducible measure machine

Theorem (Downey, Greenberg, Mihailović and Nies)

A set is low for computable measure machines

iff it is computably traceable.

Theorem (M.)

A set is low for tt-reducible measure machines

iff it is computably tt-traceable.

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A basis for tt-Schnorr randomness

Recall that A is low for ML-randomness iff

$$(\exists B) A \leq_{\mathrm{T}} B \text{ and } B \text{ is } A\text{-ML-random.}$$

Consider a set A such that

$$(\exists B) A \leq_{\mathrm{tt}} B \text{ and } B \text{ is } A\text{-tt-Schnorr random.}$$

Is it equivalent to low for tt-Schnorr randomness? .

A basis for tt-Schnorr randomness

Theorem (Franklin and Stephan 2010)

If $A \leq_{\mathrm{tt}} B$ and B is A-tt-Schnorr random, then A is Schnorr trivial.

There is a Schnorr trivial set that is not truth-table reducible to a Schnorr random set.

Theorem (Franklin and Stephan 2010)

A set A is Schnorr trivial iff $\exists B$ such that $A \leq_{\mathrm{snr}}$ and B is A-tt-Schnorr random.

Question

Question

Is there a notion such that

1. the definition uses $\leq_{\mathrm{tt}}$,
2. the definition uses tt-Schnorr randomness,
3. the notion is equivalent to Schnorr triviality,

and thus we can call a basis for tt-Schnorr randomness?

Characterization of Schnorr triviality via decidable prefix machines

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Schnorr randomness

Definition (Downey and Griffiths)

A computable measure machine is a prefix-free machine M such that $\mu([\![\mathrm{dom}(M)]\!])$ is computable.

Theorem (Downey and Griffiths)

A set A is Schnorr rnadom iff

$$K_M(A \upharpoonright n) \geq n - O(1)$$

for all computable measure machines M.

Schnorr triviality

Definition (Downey and Griffiths)

A is Schnorr reducible to B (denoted by $A \leq_{\mathrm{Sch}} B$) if each computable measure machine M there is a computable measure machine N such that

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + O(1).$$

A is Schnorr trivial if $A \leq_{\mathrm{Sch}} \emptyset$.

Decidable machines

A machine M is decidable if $\mathrm{dom}(M)$ is computable. An order is an unbounded nondecreasing function from $\mathbb{N}$ to $\mathbb{N}$.

Theorem (Bienvenu and Merkle)
A is Schnorr random iff for all decidable prefix-free machines M and computable orders g, we have

$$K_M(A \upharpoonright n) \geq n - g(n) - O(1).$$

Can we use the machines to characterize Schnorr triviality?

wdm-reducibility

Definition (M.)

A is weakly decidable prefix-free machine reducible to B (denoted by $A \leq_{\mathrm{wdm}} B$) if for each decidable prefix-free machine M and a computable order g there exists a decidable prefix-free machine N such that

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + g(n) + O(1).$$

Characterization

Theorem (M.)

A is Schnorr trivial iff $A \leq_{\mathrm{wdm}} \emptyset$.

Actually,

$$A \leq_{\mathrm{wdm}} B \iff A \leq_{\mathrm{Sch}} B$$

for all sets A, B.

Being a basis for tt-Schnorr randomness

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Similar forms

A is Schnorr trivial

iff $\forall M \exists N$ s.t. $K_N(A \restriction n) \leq K_M(n) + O(1)$

iff $\forall M, g \exists N$ s.t. $K_N(A \restriction n) \leq K_M(n) + g(n) + O(1)$

iff $\forall f \leq_{\mathrm{tt}} A \exists \{T_n\}$ s.t. $f(n) \in T_n$ for all n

iff $\forall h \exists M$ s.t. $K_M(A \restriction h(n)) < n$.

Similar form for a basis for tt-Schnorr randomness?

The proof of one implication

Recall that the following does not hold:

A is low for tt-Schnorr randomness

$\Rightarrow A \leq_{\mathrm{tt}} \exists B$ and B is A-tt-Schnorr random.

How do we prove the direction for ML-randomness:

A is low for ML-randomness

$\Rightarrow A \leq_{\mathrm{T}} \exists B$ and B is A-ML-random?

It suffices to choose a ML-random set B such that $A \leq_{\mathrm{T}} B$, which follows by the Kućera-Gács Theorem.

Space Lemma

Lemma (Merkle and Mihailović)

Given a rational $\delta > 1$ and integer $k > 0$, we can compute a length $l(\delta, k)$ such that, for any martingale d and any σ,

$$|\{\tau \in 2^{l(\delta,k)} \ : \ d(\sigma\tau) \leq \delta d(\sigma)\}| \geq k.$$

tt-Schnorr randomness

Definition (M.)

Let $d \leq_{\mathrm{tt}} A$ be a martingale.

X is A-tt-Schnorr random for d if $d(X \upharpoonright n) \leq h(n) + O(1)$ for all computable orders h.

X is A-tt-reducible random for d if $d(X \upharpoonright n) \leq O(1)$.

Then X is A-tt-Schnorr random iff X is A-tt-Schnorr random for each maritingale $d \leq_{\mathrm{tt}} A$.

A basis for tt-Schnorr randomness

Definition (M.)

A is a basis for tt-Schnorr randomness if,

for each martingale $d \leq_{\mathrm{tt}} A$ there exists a set B such that $A \leq_{\mathrm{tt}} B$ and B is A-tt-Schnorr random for d.

A is a basis for tt-reducible randomness if,

for each martingale $d \leq_{\mathrm{tt}} A$ there exists a set B such that $A \leq_{\mathrm{tt}} B$ and B is A-tt-reducible random for d.

Coincidence

Theorem (M.)

The following are equivalent for a set A:

1. A is Schnorr trivial,
2. A is a basis for tt-reducible randomness,
3. A is a basis for tt-Schnorr randomness.

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Proof sketch

Lemma

computable tt-traceable

$\Rightarrow$ a basis for tt-reducible randomness

For each Φ such that $d = \Phi^A$ is a martingale, We need to construct B such that

1. $A \leq_{\mathrm{tt}} B$,
2. $d(B \upharpoonright n) \leq O(1)$.

With the Space Lemma, construct B which has the information of A so that one can calculate d from B.

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